Solution to "Computing Subdifferentials"

Calculate the subdifferential of the following functions:

Part (a): $f(x) = \max\{1, |x-1|\}$ on $\mathbb{R}$.
Equivalently, express $f(x) = \max\{1-x, 1, x-1\}$ in piece-wise form:

$$f(x) = \begin{cases} 1-x & \text{for } x < 0 \\ 1 & \text{for } 0 \leq x < 2 \\ x-1 & \text{for } x \geq 2 \end{cases}$$

The functions $f_1(x) = 1-x$ and $f_2(x) = 1$ and $f_3(x) = x-1$ all are proper, lower semi-continuous, and convex. Thus, each of the three is also differentiable across its domain.

It is evident that $I(0) = \{1,2\}$ and $I(2) = \{2,3\}$.

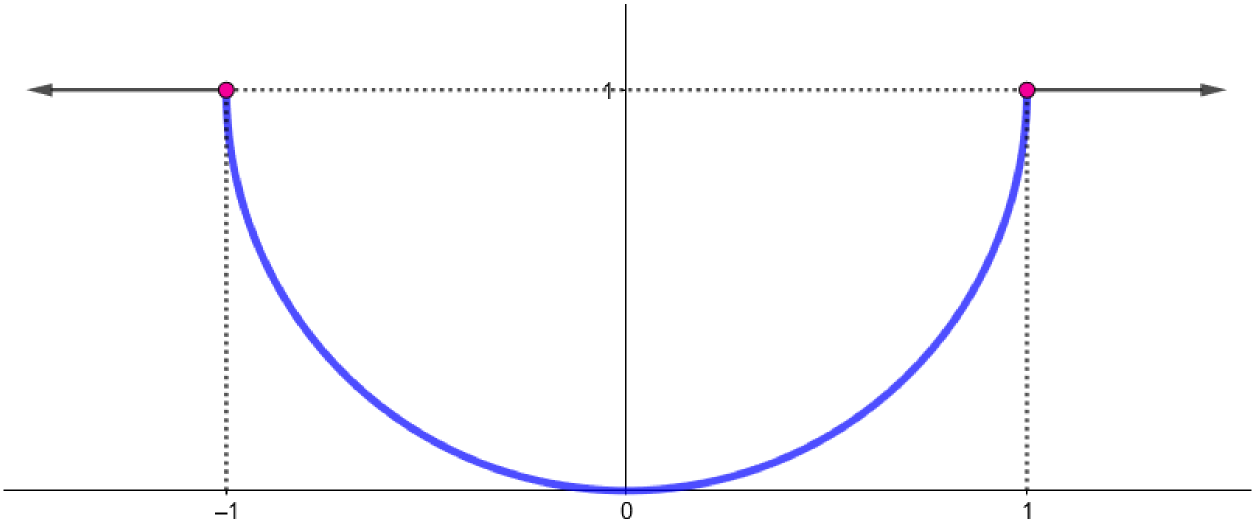
Therefore, $\partial f(0) = \operatorname{conv}\big(\partial f_1(0) \cup \partial f_2(0)\big) = \operatorname{conv}\big(\{-1, 0\}\big) = [-1, 0]$.
Similarly, $\partial f(2) = \operatorname{conv}\big(\partial f_2(2) \cup \partial f_3(2)\big) = \operatorname{conv}\big(\{0, 1\}\big) = [0, 1]$.

Finally, the subdifferential of f is:

$$\partial f(x) = \begin{cases} \{-1\} & \text{for } x < 0 \\ [-1, 0] & \text{for } x = 0 \\ \{0\} & \text{for } 0 < x < 2 \\ [0, 1] & \text{for } x = 2 \\ \{1\} & \text{for } x > 2 \end{cases}$$

Part (b): $f(x) = 1 - \sqrt{1-x^2}$ if $x \in [-1,1]$ and ∞ otherwise.

The function f is plotted below for $x \in \text{dom}(f) = [-1,1]$.



It is evident that $N_{\text{epi}(f)}(-1,1) = \{(\zeta,0) : \zeta \le 0\}$ and $N_{\text{epi}(f)}(1,1) = \{(\zeta,0) : \zeta \ge 0\}$.
Since $\nexists(\zeta,-1) \in N_{\text{epi}(f)}(-1,1) \cup N_{\text{epi}(f)}(1,1)$, then predict $\partial f(-1) = \emptyset$ and $\partial f(1) = \emptyset$.

Proof:
At the point $(1,1)$: Let $\zeta \in \partial f(1)$.
Then for all $x \in \text{dom}(f) = [-1,1]$ obtain $f(x) \ge f(x_0) + \langle \zeta, x - x_0 \rangle \implies \zeta(1-x) \ge \sqrt{1-x^2}$.
Thus, $\zeta \ge \sqrt{1-x^2}/(1-x)$ for $x \in [-1,1[$.

Let $g(x) = \sqrt{1-x^2}/(1-x)$ for $x \in [-1,1[$.
Here, g is a strictly increasing function since $g'(x) = 1/[(1-x)\sqrt{1-x^2}] > 0$.

Therefore, the greatest lower bound of ζ for all $x \in \text{dom}(f)$ is $\lim_{x \to 1^-} g(x) = \infty$.
Since $\zeta \in \mathbb{R}$, then $\nexists \zeta$ for which the inequality is met for all $x \in [-1,1]$.

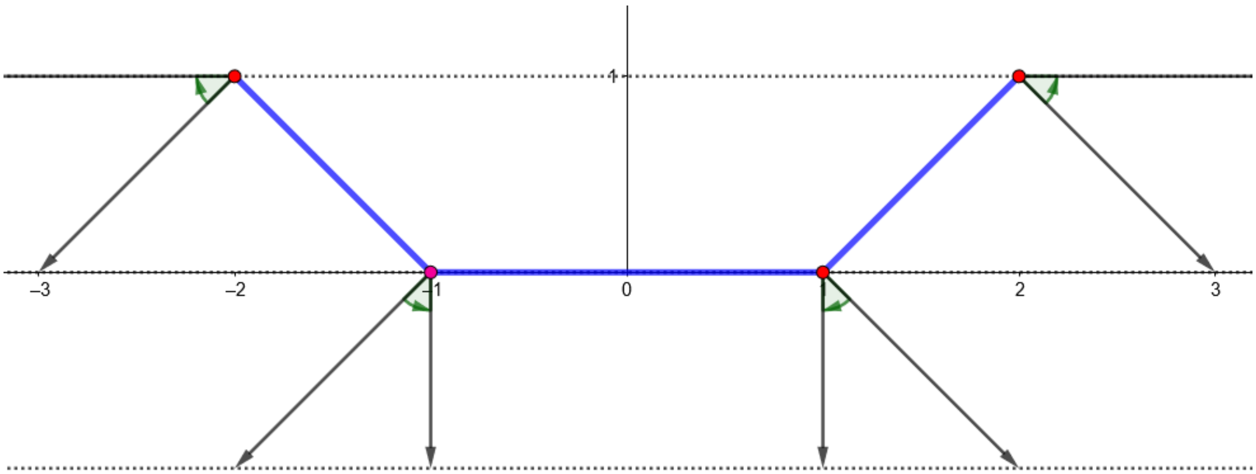
As such, $\partial f(1) = \emptyset$, and by symmetry $\partial f(-1) = \emptyset$.

The subgradients of f are defined on its effective domain.
Here, f is differentiable on $]-1,1[$ and thus:

$$\partial f(x) = \begin{cases} \left\{ \dfrac{x}{\sqrt{1-x^2}} \right\} & \text{for } -1 < x < 1 \\ \emptyset & \text{for } x \in \{-1,1\} \end{cases}$$

Part (c): $f(x) = \begin{cases} 0 & x \in [-1, 1] \\ |x| - 1 & x \in [-2, -1[\cup]1, 2] \\ \infty & x \in]-\infty, -2[\cup]2, \infty[\end{cases}$

The function f is plotted below for $x \in \mathrm{dom}(f) = [-2, 2]$.



On its effective domain, f can be rewritten as: $f(x) = \max\{-x - 1, 0, x - 1\}$.

The functions $f_1(x) = -x - 1$ and $f_2(x) = 0$ and $f_3(x) = x - 1$ all are proper, lower semi-continuous, convex, and each is also differentiable across its domain.

It is evident that $I(-1) = \{1, 2\}$ and $I(1) = \{2, 3\}$.

Therefore, $\partial f(-1) = \mathrm{conv}\big(\partial f_1(-1) \cup \partial f_2(-1)\big) = \mathrm{conv}\big(\{-1, 0\}\big) = [-1, 0]$.
Similarly, $\partial f(1) = \mathrm{conv}\big(\partial f_2(1) \cup \partial f_3(1)\big) = \mathrm{conv}\big(\{0, 1\}\big) = [0, 1]$.

At the point $(2, 1)$: $f(x) \geq f(x_0) + \langle \zeta, x - x_0 \rangle = 1 + \zeta(x - 2)$
Restrict the study on the $\mathrm{dom}(f)$ since the inequality requires the function to be proper.

***Case 1*:** For $-1 \leq x \leq 1$: then $f(x) = 0$ and $-3 \leq x - 2 \leq -1$
Since $1/3 \leq -1/(x - 2) \leq 1$ then $0 \geq 1 + \zeta(x - 2) \implies \zeta \geq -1/(x - 2) \implies \zeta \geq 1$.

***Case 2*:** For $1 < x < 2$: then $f(x) = x - 1$ and $-1 \leq x - 2 < 0$
Since $x - 1 \geq 1 + \zeta(x - 2) \implies x - 2 \geq \zeta(x - 2) \implies \zeta \geq 1$.

***Case 3*:** For $-2 \leq x < -1$: then $f(x) = 1 - x$ and $-4 \leq x - 2 \leq -3$
Since $1 - x \geq 1 + \zeta(x - 2) \implies \zeta \geq -1 - 2/(x - 2) \implies \zeta \geq -1/3$.

Since the inequality should hold $\forall x \in \mathrm{dom}(f)$, then select the restriction $\zeta \geq 1$.

Conclude $\partial f(2) = [1, \infty[$ and by symmetry, $\partial f(-2) =]-\infty, -1]$.

Finally, the subdifferential of f is:

$$\partial f(x) = \begin{cases}]-\infty, -1] & \text{for } x = -2 \\ \{-1\} & \text{for } -2 < x < -1 \\ [-1, 0] & \text{for } x = -1 \\ \{0\} & \text{for } -1 < x < 1 \\ [0, 1] & \text{for } x = 1 \\ \{1\} & \text{for } 1 < x < 2 \\ [1, \infty[& \text{for } x = 2 \end{cases}$$