Solution to "QP Optimality Condition"

We consider the following quadratic programming problem:

$$(Q)\begin{cases}\min & f(x) = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + \alpha x_1 + \beta x_2 + \gamma x_3 \\ \text{s.t.} & x_1 + 2x_2 + 3x_3 \leq 10 \text{ and } 3x_1 + 2x_2 + x_3 \leq 14.\end{cases}$$

Prove that (Q) has a minimal solution if and only if $\alpha + \beta + \gamma \leq 0$.

Rewrite Q in the following form:

$$(Q)\begin{cases}\min & f(x) = \langle Wx, x\rangle + \langle a, x\rangle \\ \text{s.t.} & Ax \leq b,\end{cases}$$

where $a \in \mathbb{R}^3$, W is a symmetric positive semidefinite matrix, A is a 2×3 matrix and $b \in \mathbb{R}^2$.

Propose the following vectors and matrices:

$$W = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

Compute the matrix products,

$$Wx = \begin{bmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 2x_3 \end{bmatrix}, \quad Ax = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 3x_1 + 2x_2 + x_3 \end{bmatrix}.$$

Validate the choice of W,

$$\begin{aligned}\langle Wx, x\rangle &= 2x_1^2 - x_1x_2 - x_1x_3 + 2x_2^2 - x_2x_3 - x_2x_1 + 2x_3^2 - x_3x_1 - x_3x_2 \\ &= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_1x_3 \\ &= x_1^2 - 2x_1x_2 + x_2^2 + x_2^2 - 2x_2x_3 + x_3^2 + x_3^2 - 2x_1x_3 + x_1^2 \\ &= (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \\ &= f(x) - \langle a, x\rangle.\end{aligned}$$

The constraints $x_1 + 2x_2 + 3x_3 \leq 10$ and $3x_1 + 2x_2 + x_3 \leq 14$ are clearly equivalent to $Ax \leq b$.

Compute the eigenvalues of W,

$$\begin{aligned}\det(W - \lambda I) &= \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^3 - 3(2-\lambda) - 2 \\ &= -\lambda^3 + 6\lambda^2 - 9\lambda \\ &= -\lambda(\lambda - 3)^2 = 0\end{aligned}$$

The eigenvalues of W are $\lambda_1 = \lambda_2 = 3 \geq 0$ and $\lambda_3 = 0 \geq 0$.
Therefore W is positive semidefinite.

Use the following assertion,

$$(Q) \text{ attains a minimum.} \iff \langle a, v \rangle \geq 0 \;\; \forall v : Av \leq 0 \text{ and } v \in \ker W.$$

Compute the kernel of W,

$$Wv = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.$$

Since W is non-invertible ($\det W = 0$), set $v_1 = t \in \mathbb{R}$ to obtain $v_1 = v_2 = v_3 = t$. Thus

$$\ker W = \left\{ t \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}} : t \in \mathbb{R} \right\}$$

Compute Av,

$$Av = t \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 6t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq 0 \Longrightarrow t \leq 0.$$

Obtain $\langle a, v \rangle \geq 0 \iff t(\alpha + \beta + \gamma) \geq 0$.
Since $t \leq 0$, thus $\alpha + \beta + \gamma \leq 0$, as desired.