Solution to "Convexity & Optimality"

Let $S \subset \mathbb{R}^2$ be the set defined by

$$S := \{(x, y) : x \geq 0, y \geq 0, -x + y \leq 2 \text{ and } 2x + 3y \leq 11\}.$$

We define on the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) := x^2 + y^2 - 8x - 20y + 89.$$

Part (a): Prove that S and f are convex.
Let $(x_1, y_1), (x_2, y_2) \in S$ and let $X = \lambda x_1 + (1-\lambda)x_2$ and $Y = \lambda y_1 + (1-\lambda)y_2$ for all $\lambda \in [0, 1]$.
It is clear, using convex combinations, that:

$$\begin{aligned} x_1, x_2, y_1, y_2 \geq 0 &\Longrightarrow X, Y \geq 0 \\ -x_1 + y_1 \leq 2 \text{ and } -x_2 + y_2 \leq 2 &\Longrightarrow -X + Y \leq 2 \\ 2x_1 + 3y_1 \leq 11 \text{ and } 2x_2 + 3y_2 \leq 11 &\Longrightarrow 2X + 3Y \leq 11 \end{aligned}$$

Therefore $(X, Y) \in S$ meaning that $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in S$, thus S is convex.

Compute the Hessian matrix of f (it is differentiable):

$$\nabla^2 f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$$

Its eigenvalues are $\lambda_1 = \lambda_2 = 2 > 0$ and thus f is strictly convex.

Part (b): Prove that $(1, 3)$ is an optimal solution for minimizing f over S.
It is sufficient to prove that:

$$\langle z - z^*, \nabla f(z^*) \rangle \geq 0, \ \forall z \in S$$

Where $z = (x, y)$, $z^* = (1, 3)$, and $\nabla f(z^*) = (2x^* - 8, 2y^* - 20) = (-6, -14)$.

Obtain $\langle z - z^*, \nabla f(z^*) \rangle = (x - 1, y - 3) \cdot (-6, -14) = -6x - 14y + 48$.
Rearrange $-6x - 14y + 48 = (2x - 2y + 4) + (-8x - 12y + 44) = 2(x - y + 2) + 4(-2x - 3y + 11)$.
Since $z \in S$, then $\langle z - z^*, \nabla f(z^*) \rangle = 2\underbrace{(x - y + 2)}_{\geq 0} + 4\underbrace{(-2x - 3y + 11)}_{\geq 0} \geq 0$.

Therefore $(1, 3)$ minimizes f over S.

Part (c): Prove that $(1, 3)$ is the only optimal solution.
Since f is proven in Part (a) to be strictly convex, and that is clearly proper, then any global minimum is the only global minimum.