Solution to "Conjugate Functions and Subdifferentials"

Let $f : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ be a proper, closed and convex function such that

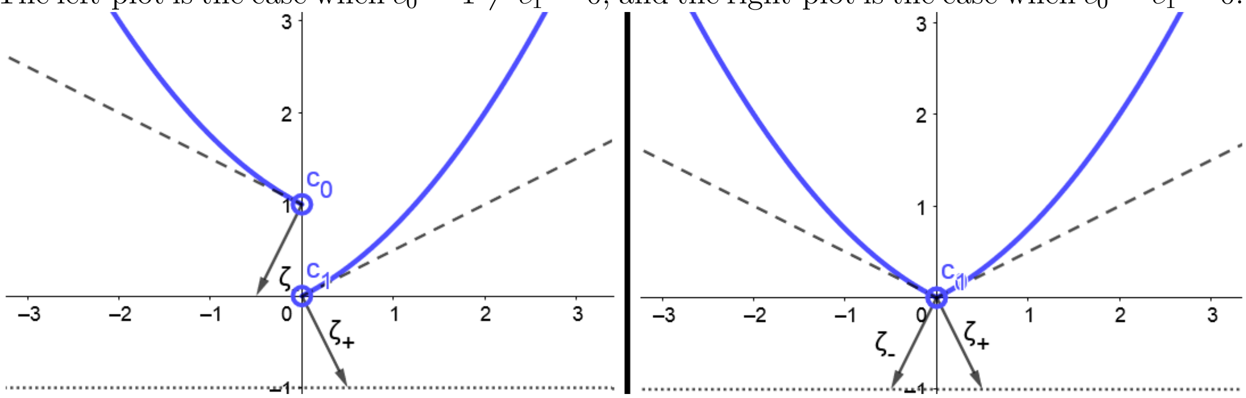
$$\partial f(x) = \begin{cases} \frac{1}{2}(x-1) & x < 0, \\ \left[-\frac{1}{2}, \frac{1}{2}\right] & x = 0, \\ \frac{1}{2}(x+1) & x > 0. \end{cases}$$

Find f, ∂f^* and f^*.

Part 1: f
From the properties of f, deduce that f is differentiable at $x \neq 0$.
Compute the antiderivatives of $\partial f(x) = \{f'(x)\}$ for all $x \neq 0$.

For $x < 0 : f(x) = \frac{1}{4}(x^2 - 2x) + c_0$, and for $x > 0 : f(x) = \frac{1}{4}(x^2 + 2x) + c_1$. Here c_0 and c_1 are arbitrary constants not necessarily equal, as f is not necessarily continuous at 0.

The left-plot is the case when $c_0 = 1 \neq c_1 = 0$, and the right-plot is the case when $c_0 = c_1 = 0$.



***Lemma*:** $c_0 = c_1 = c$ and f is continuous at $x = 0$.
Prove by contradiction: Assume $c_0 \neq c_1$ and thus $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$. Imply from its piecewise definition that f is discontinuous and undefined at 0. Thus $0 \notin \text{dom}(f)$. However, $\partial f(x)$ is only defined for all $x \in \text{dom}(f)$. Since $\partial f(0)$ exists, then $0 \in \text{dom}(f)$.
Contradiction.

Conclude that $c_0 = c_1 = c$. And thus:

$$f(x) = \begin{cases} \frac{1}{4}(x^2 - 2x) + c & x < 0, \\ c & x = 0, \\ \frac{1}{4}(x^2 + 2x) + c & x > 0. \end{cases}$$

Equivalently, $f(x) = \frac{1}{4}\big(x^2 + 2|x|\big) + c$, for all $c \in \mathbb{R}$.

Part 2: ∂f^*
Since f is proper, closed, and convex, therefore: $y \in \partial f(x) \iff x \in \partial f^*(y)$.

***Case 1*:** $x < 0$
Here $y = \frac{1}{2}(x-1) < -\frac{1}{2} \iff x = 2y + 1 < 0$.

***Case 2*:** $x > 0$
Here $y = \frac{1}{2}(x+1) > \frac{1}{2} \iff x = 2y - 1 > 0$.

***Case 3*:**: $x = 0$
Here $y \in \big[-\frac{1}{2}, \frac{1}{2}\big] \iff x = 0$.

Conclude from the cases above that:

$$\partial f^*(y) = \begin{cases} 2y+1 & y < -\frac{1}{2}, \\ 0 & y \in \big[-\frac{1}{2}, \frac{1}{2}\big], \\ 2y-1 & y > \frac{1}{2}. \end{cases}$$

Part 3: f^*
Since f is proper, closed and convex, obtain that: $f^*(y) = \langle x, y\rangle - f(x) = xy - f(x)$.

***Case 1*:** $x < 0 \iff y < -\frac{1}{2}$
Here $f(x) = \frac{1}{4}(x^2 - 2x) + c$ and as above $x = 2y + 1$.
Replace in $f^*(y) = xy - f(x)$ to obtain: $f^*(y) = y^2 + y + \frac{1}{4} - c$.

***Case 2*:** $x > 0 \iff y > \frac{1}{2}$
Here $f(x) = \frac{1}{4}(x^2 + 2x) + c$ and as above $x = 2y - 1$.
Replace in $f^*(y) = xy - f(x)$ to obtain: $f^*(y) = y^2 - y + \frac{1}{4} - c$.

***Case 3*:** $x = 0 \iff y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$
Here $f(x) = c$ and as above $x = 0$.
Replace in $f^*(y) = xy - f(x)$ to obtain: $f^*(y) = -c$.

Conclude from the cases above that:

$$f^*(y) = \begin{cases} y^2 + y + \frac{1}{4} - c & y < -\frac{1}{2}, \\ -c & y \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ y^2 - y + \frac{1}{4} - c & y > \frac{1}{2}. \end{cases}$$

For the same arbitrary constant c as in Part 1.

Below is plotted the conjugate function $f^*(y)$ for the case where $c = 1$.