Given the LTI system $\begin{cases}\dot{\mathbf{x}} = \mathrm{A}\mathbf{x} + \mathrm{B}\mathbf{u} \\ \mathbf{y} = \mathrm{C}\mathbf{x}\end{cases}$ with:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}, \qquad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathrm{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Assumptions

- Output value $\mathbf{y}(t)$ is **available** for feedback control.
- State value $\mathbf{x}(t)$ is **unavailable**.

Luenberger Observer Design

The Luenberger observer of the form $\begin{cases}\dot{\hat{\mathbf{x}}} = \mathrm{A}\hat{\mathbf{x}} + \mathrm{B}\mathbf{u} + \mathrm{L}(\mathbf{y} - \hat{\mathbf{y}}) \\ \hat{\mathbf{y}} = \mathrm{C}\mathbf{x}\end{cases}$ inherits the value of the matrices $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ from the LTI system mentioned above.

The gain matrix $\mathrm{L} \in \mathbb{R}^{3\times 1}$ is designed such that $\mathrm{A} - \mathrm{LC}$ has eigenvalues $\lambda \in \{-1, -2, -3\}$.

MATLAB Solution

Define the system matrices.

```
%% MATRIX DEFINITIONS
>> A = [0 1 0; 0 1 1; 1 -2 1]; C = [0 0 1];
```

Check if the system is fully-observable.

```
%% OBSERVABILITY
>> rank(obsv(A,C))
ans = 3
```

Since $\dim \mathbf{x} = 3$ therefore the system is **observable**.

Design the gain matrix of the Luenberger Observer using `place`. ($\lambda \in \{-1, -2, -3\}$)

```
%% POLE PLACEMENT
>> L = place(A',C', [-1, -2, -3])'
L =
  -30.0000
  -23.0000
    8.0000
```

Thus, $\boxed{\mathrm{L} = \begin{bmatrix} -30 \\ -23 \\ 8 \end{bmatrix}}$.

Verification step, check the eigenvalues of $\mathrm{A} - \mathrm{LC}$.

```
>> eigs(A - L*C)'
ans =
    -1.0000   -2.0000   -3.0000
```

Effectively, the desired eigenvalues are $\lambda \in \{-1, -2, -3\}$.

Linear Observer Dynamic Equations

The observer equations are,

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \mathrm{A}\hat{\mathbf{x}} + \mathrm{B}\mathbf{u} + \mathrm{L}(\mathbf{y} - \hat{\mathbf{y}}) \\ \hat{\mathbf{y}} = \mathrm{C}\hat{\mathbf{x}} \end{cases}$$

All matrices have been determined. Replace all values to obtain,

$$\boxed{\begin{cases} \dot{\hat{\mathbf{x}}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \hat{\mathbf{x}} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{u} + \begin{bmatrix} -30 \\ -23 \\ 8 \end{bmatrix} (\mathbf{y} - \hat{\mathbf{y}}) \\ \hat{\mathbf{y}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \hat{\mathbf{x}} \end{cases}}$$

Feedback Control Design

The linear feedback gain $\mathrm{K} \in \mathbb{R}^{1\times 3}$ will be designed such that for input $\mathbf{u} = \mathrm{K}\mathbf{x}$ the eigenvalues of the closed-loop system are $\lambda \in \{-1, -2, -3\}$.

MATLAB Solution

Define the system matrices.

```
%% MATRIX DEFINITIONS
>> A = [0 1 0; 0 1 1; 1 -2 1]; B = [0;0;1];
```

Check the system's asymptotic stability.

```
%% STABILITY
>> real(eig(A))'
ans =
    0.4302    0.7849    0.7849
```

Since the eigenvalues have positive real parts, the system $\{\mathrm{A}, \mathrm{B}\}$ is **unstable**.

Check if the system is fully-controllable.

```
%% CONTROLLABILITY
>> r = rank(ctrb(A,B))
r = 3
```

Since $\dim \mathbf{x} = 3$ therefore, the system $\{\mathrm{A}, \mathrm{B}\}$ is **controllable**.

Design the feedback gain matrix $\mathrm{K} \in \mathbb{R}^{1\times 3}$ such that $\lambda \in \{-1, -2, -3\}$.

```
%% POLE PLACEMENT
>> K = -place(A,B,[-1,-2,-3])
K =
    -7.0000  -16.0000   -8.0000
```

Thus, $\boxed{\mathrm{K} = \begin{bmatrix} -7 & -16 & -8 \end{bmatrix}}$.

The controller equation is $\mathbf{u}(t) = \mathrm{K}\mathbf{x}(t)$.

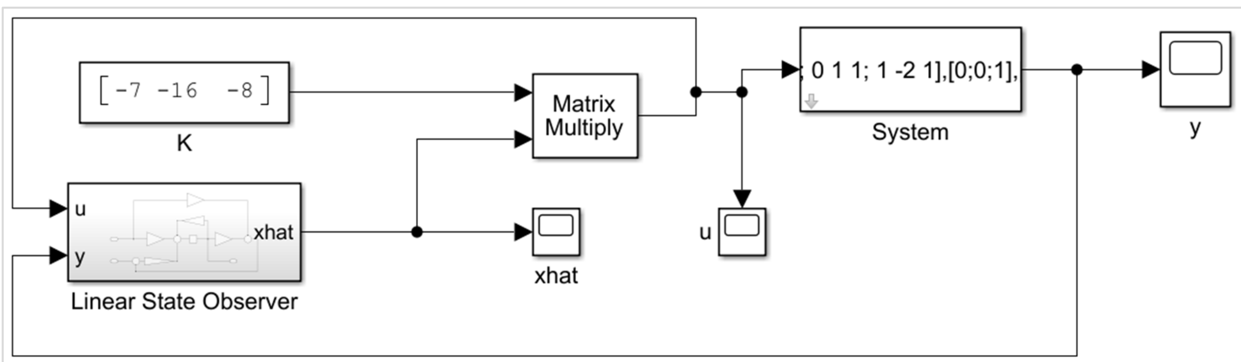
$$\mathbf{u}(t) = \begin{bmatrix} -7 & -16 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -7x_1 - 16x_2 - 8x_3$$

Finally, obtain $\boxed{\mathbf{u}(t) = -7x_1 - 16x_2 - 8x_3}$

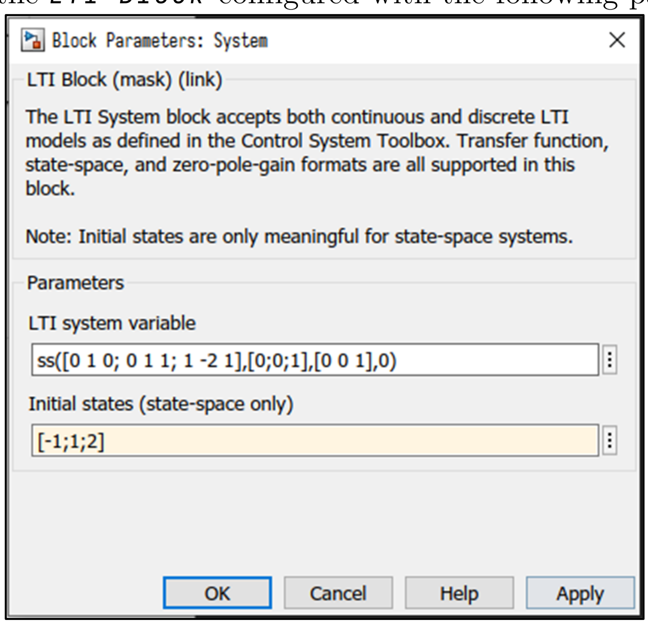
Simulink Implementation

Refer to ***luenberger.slx*** *for the implementation in* ***Simulink***.

The output-feedback controller has been implemented in ***Simulink*** as follows.

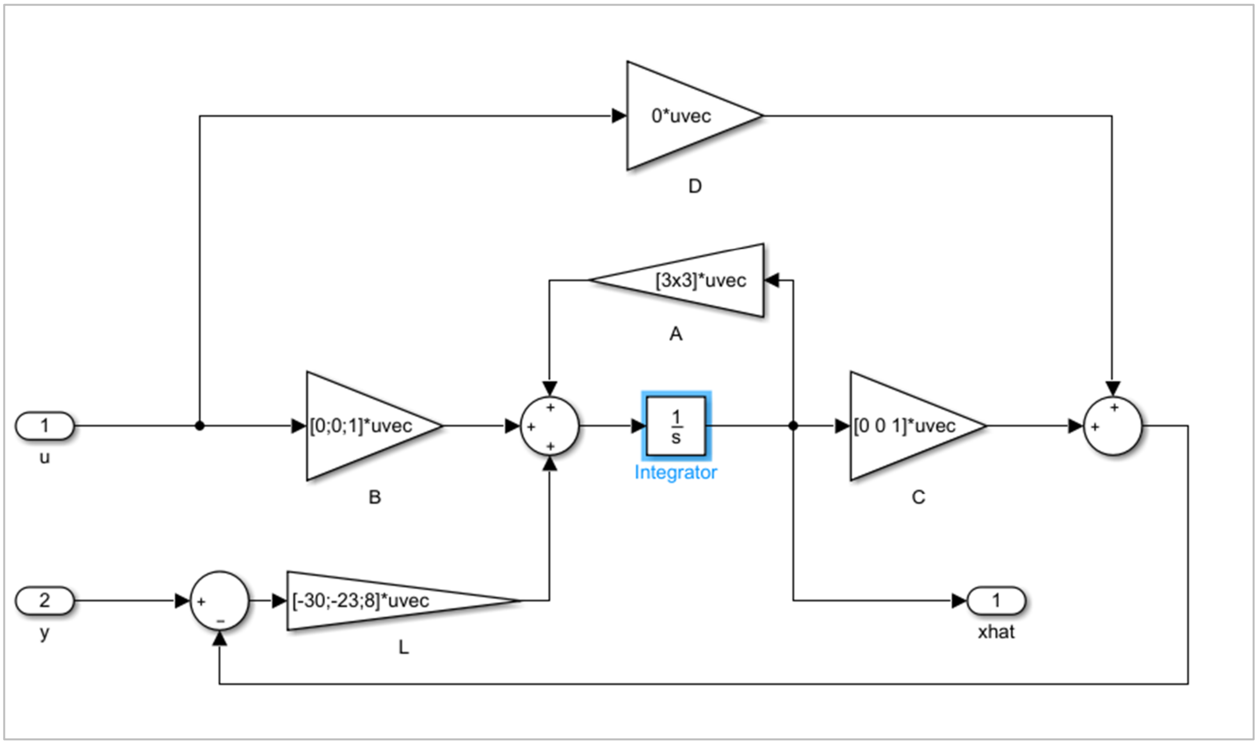


The block ***System*** is the ***LTI Block*** configured with the following parameters.



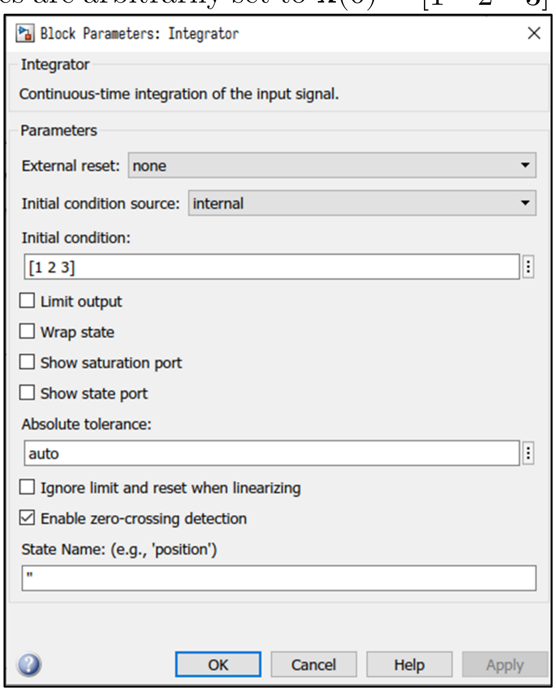
The initial states were arbitrarily set to $\mathbf{x}(0) = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^\top$.

The block ***Linear State Observer*** is a subsystem containing the following:



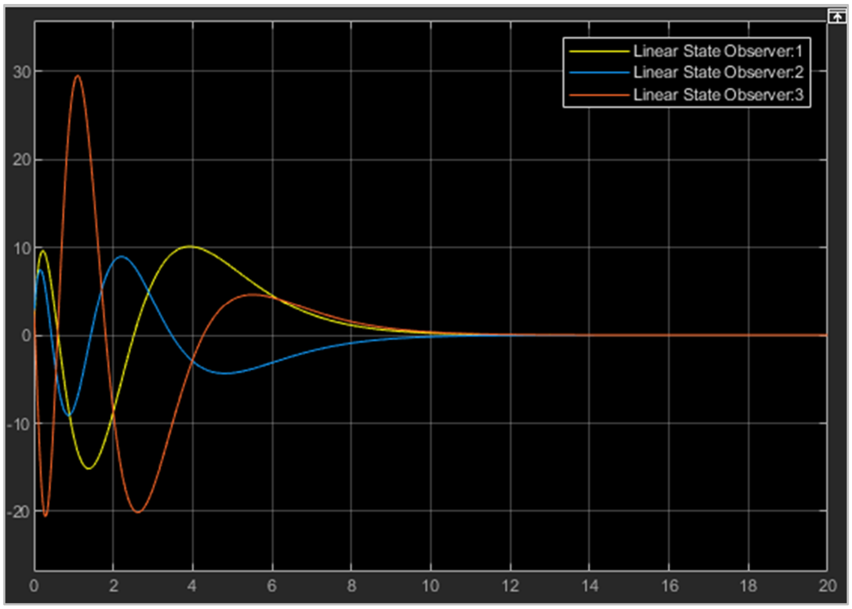
This is the implementation for the dynamical system that describes the Luenberger observer developed earlier. The matrix values in this system are all listed above.

The initial estimated states are arbitrarily set to $\hat{\mathbf{x}}(0) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ via the `Integrator` block.

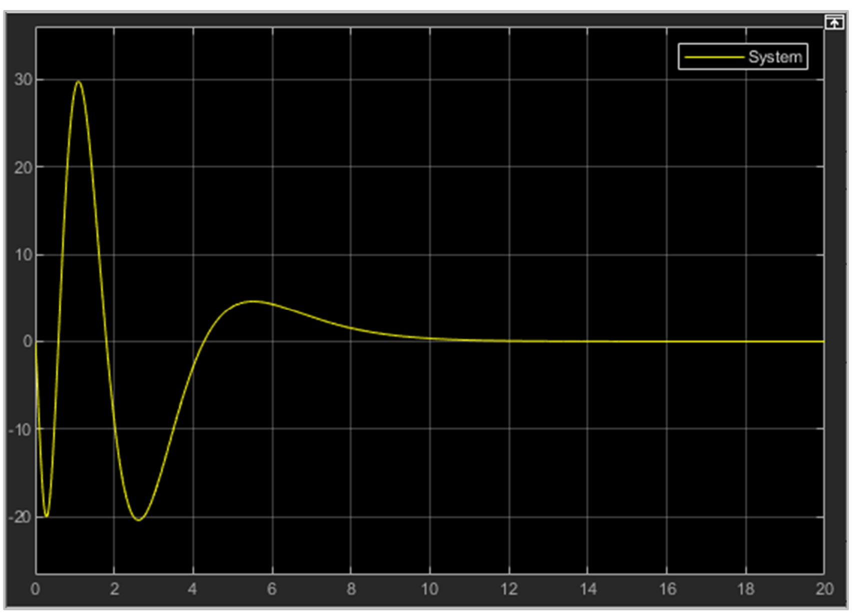


Plots

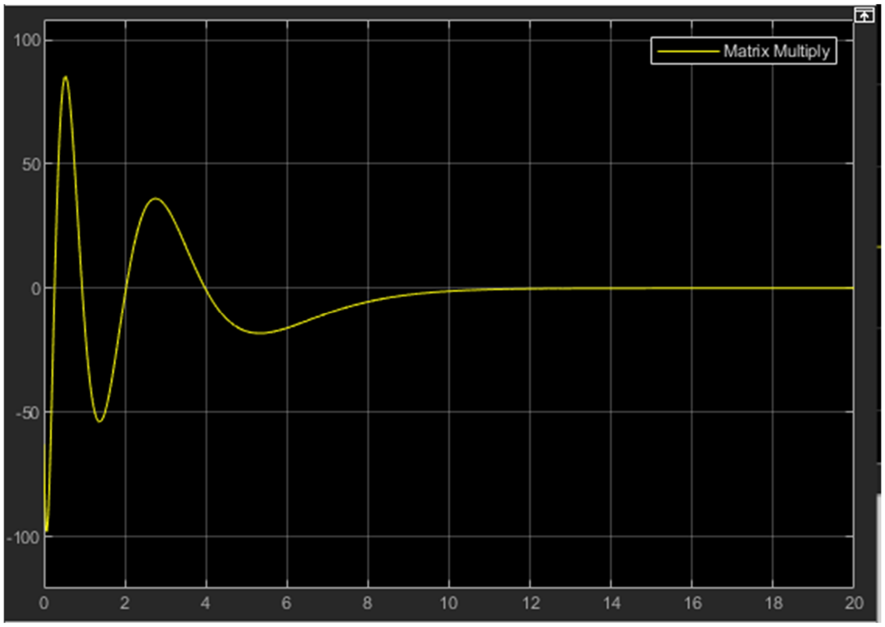
This first plot is for the observer states i.e. the estimated states $\hat{\mathbf{x}} = [\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3]^\top$.



This second plot is for the output of the LTI system $\mathbf{y} = \mathrm{C}\mathbf{x}$.



This third plot is for the feedback input $\mathbf{u} = \mathrm{K}\hat{\mathbf{x}}$.



All three plots show that the system asymptotically stabilizes while reaching its goal.