Solution to "Subdifferential of Distance Function"

Let $C \subset \mathbb{R}^n$ be a nonempty, closed and convex set. We denote by $d_C(\cdot)$ the distance function associated to C. Find the subdifferential $\partial d_C(x)$ for all $x \in C$.

Proof:

Let x_0 be a fixed point in C. The distance function $d_C(x) = \inf_{c \in C} \|x - c\|$ over a convex set C is convex. Moreover, it is clearly proper and lower semi-continuous.

With these properties verified, assert the following with $d_C(x_0) = 0$

$$\partial d_C(x_0) = \left\{\zeta \in \mathbb{R}^n : d_C(x) \geq \langle \zeta, x - x_0 \rangle,\ \forall x \in \mathbb{R}^n\right\}$$

***Lemma 1*:** $\partial d_C(x_0) \subset N_C(x_0)$

Let $\zeta \in \partial d_C(x_0)$, then obtain $d_C(x) \geq \langle \zeta, x - x_0 \rangle$ for all $x \in \mathbb{R}^n$.

Then for all $x \in C$, obtain $d_C(x) = 0$, and thus that $\langle \zeta, x - x_0 \rangle \leq 0$.

This gives $\zeta \in N_C(x_0)$. Therefore, $\partial d_C(x_0) \subset N_C(x_0)$.

***Lemma 2*:** $\partial d_C(x_0) \subset \bar{B}$

Let $\zeta \in \partial d_C(x_0)$, then obtain $d_C(x) \geq \langle \zeta, x - x_0 \rangle$ for all $x \in \mathbb{R}^n$.

The function $d_C(x)$ is 1-Lipschitz, as such: $|d_C(x_1) - d_C(x_2)| \leq 1 \cdot \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^n$.

Choose $x_1 = x \in \mathbb{R}^n$ and $x_2 = x_0 \in C$ to obtain $\|x - x_0\| \geq d_C(x)$ for all $x_1, x_2 \in \mathbb{R}^n$.

Combine this with the previous result to obtain $\|x - x_0\| \geq \langle \zeta, x - x_0 \rangle$ for all $x \in \mathbb{R}^n$.

Now choose $x = \zeta + x_0$ to obtain $\|\zeta\| \geq \langle \zeta, \zeta \rangle$ and thus $\|\zeta\| \leq 1$.

As such $\zeta \in \bar{B}$, giving $\partial d_C(x_0) \subset \bar{B}$.

***Lemma 3*:** $\bar{B} \cap N_C(x_0) \subset \partial d_C(x_0)$

Prove by contradiction: Let $\zeta \in N_C(x_0) \cap \bar{B}$.

Assume $\zeta \notin \partial d_C(x_0)$, then $\exists y \in \mathbb{R}^n : \langle \zeta,\ y - x_0 \rangle > d_C(y) = \|y - \bar{c}\|$, where $\bar{c} = \mathrm{proj}_C(y)$.

Since $\zeta \in N_C(x_0)$ then $\langle \zeta, x - x_0 \rangle \leq 0$ for all $x \in C$. In particular, $\bar{c} \in C$, thus $\langle \zeta, x_0 - \bar{c} \rangle \geq 0$.

Add the inequalities: $0 + \|y - \bar{c}\| < \langle \zeta, x_0 - \bar{c} \rangle + \langle \zeta,\ y - x_0 \rangle = \langle \zeta, y - \bar{c} \rangle \leq \|\zeta\| \cdot \|y - \bar{c}\|$.

Since $\zeta \in \bar{B}$, then $\|\zeta\| \leq 1$ and thus: $\|y - \bar{c}\| < \langle \zeta, y - \bar{c} \rangle \leq \|y - \bar{c}\|$.

Contradiction.

***Conclusion*:**

Conjunct the proven Lemmas to obtain $\partial d_C(x_0) = \bar{B} \cap N_C(x_0)$.