Solution to "Quadratic Programming"

We consider the quadratic programming problem:

$$(P)\begin{cases}\min & f(x) = (x_1 - x_2)^2 + (x_2 + x_3)^2 + (x_3 - x_1)^2 \\ \text{s.t.} & x_1 + 2x_2 + 3x_3 = 10 \text{ and } 3x_1 + 2x_2 + x_3 = 14.\end{cases}$$

Part (a): Find a vector $b \in \mathbb{R}^2$, a symmetric matrix W of size 3×3 and a matrix A of size 2×3 such that

$$f(x) = \frac{1}{2}\langle Wx, x\rangle \text{ and the constraint is } Ax = b.$$

Propose the following vectors and matrices:

$$W = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

Compute the matrix products,

$$Wx = \begin{bmatrix} 4x_1 - 2x_2 - 2x_3 \\ -2x_1 + 4x_2 + 2x_3 \\ -2x_1 + 2x_2 + 4x_3 \end{bmatrix}, \quad Ax = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 3x_1 + 2x_2 + x_3 \end{bmatrix}.$$

Validate the choice of W,

$$\begin{aligned}\frac{1}{2}\langle Wx, x\rangle &= \frac{1}{2}\Big(4x_1^2 - 2x_1x_2 - 2x_1x_3 + 4x_2^2 + 2x_2x_3 - 2x_1x_2 + 2x_2x_3 + 4x_3^2\Big) \\ &= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + 2x_2x_3 - 2x_1x_3 \\ &= x_1^2 - 2x_1x_2 + x_2^2 + x_2^2 + 2x_2x_3 + x_3^2 + x_3^2 - 2x_1x_3 + x_1^2 \\ &= (x_1 - x_2)^2 + (x_2 + x_3)^2 + (x_3 - x_1)^2 \\ &= f(x).\end{aligned}$$

The constraints $x_1 + 2x_2 + 3x_3 = 10$ and $3x_1 + 2x_2 + x_3 = 14$ are clearly equivalent to $Ax = b$.

Part (b): Verify that W is positive definite and that the rank of A is 2.

Compute the eigenvalues of W,

$$\begin{aligned}
\det(W-\lambda I) &= \begin{vmatrix} 4-\lambda & -2 & -2 \\ -2 & 4-\lambda & 2 \\ -2 & 2 & 4-\lambda \end{vmatrix} \\
&= (4-\lambda)^3 - 12(4-\lambda) + 16 \\
&= -\lambda^3 + 12\lambda^2 - 48\lambda + 64 - 48 + 12\lambda + 16 \\
&= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 \\
&= -\lambda^3 + 4\lambda^2 - 4\lambda + 8\lambda^2 - 32\lambda + 32 \\
&= -\lambda(\lambda^2 - 4\lambda + 4) + 8(\lambda^2 - 4\lambda + 4) \\
&= (8-\lambda)(\lambda-2)^2 = 0
\end{aligned}$$

The eigenvalues of W are $\lambda_1 = \lambda_2 = \lambda_{1,2} = 2 \geq 0$ and $\lambda_3 = 8 \geq 0$.
Therefore W is positive definite.

Reduce A to row echelon form,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + (1/2)R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -4 & -8 \end{bmatrix}$$

The rows of A are clearly linearly independent, and therefore $\text{rank}(A) = 2$.

Part (c): Diagonalize the matrix W and deduce its square root $W^{1/2}$.

Compute the eigenvectors of W for $\lambda_{1,2} = 2$,

$$(W - \lambda_{1,2} I)v^{(1,2)} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Since this eigenvalue is a double-solution, set two parameters $v_2 = s$, and $v_3 = t$.
Obtain $2v_1 - 2v_2 - 2v_3 = 0$, giving $v_1 = s + t$.

$$v^{(1,2)} = \begin{bmatrix} s+t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Compute the eigenvector of W for $\lambda_3 = 8$,

$$(W - \lambda_3 I)v^{(3)} = \begin{bmatrix} -4 & -2 & -2 \\ -2 & -4 & 2 \\ -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

Set $v_2 = t$ to obtain $-4v_1 - 2t - 2v_3 = 0$ and $-2v_1 - 4t + 2v_3 = 0$.
Add these two equations to obtain $-6v_1 - 6t = 0 \Longrightarrow v_1 = -t$.
Subtract them to obtain $6t - 6v_3 = 0 \Longrightarrow v_3 = t$.

$$v^{(3)} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The matrix W is diagonalizable since its algebraic multiplicities $m_{\lambda_{1,2}} = 2$ and $m_{\lambda_3} = 1$ are equal to its geometric multiplicities $M_{\lambda_{1,2}} = \dim v^{(1,2)} = 2$ and $M_{\lambda_3} = \dim v^{(3)} = 1$.

It can be written as $W = VDV^{-1}$ with $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \operatorname{diag}(2, 2, 8)$ and $V = \begin{bmatrix} v^{(1)} & v^{(2)} & v^{(3)} \end{bmatrix}$. Use $V^{-1} = (\det V)^{-1}[C_V]^\mathsf{T}$ where $c_{ij} = (-1)^{i+j} m_{ij}$ (Cofactor Method). Compute the determinant,

$$\det V = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -3.$$

Compute the cofactor matrix, and thus obtain the inverse of V,

$$C_V = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 1 & -1 \\ 1 & -2 & -1 \end{bmatrix} \Longrightarrow V^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

It is now possible to compute the square root matrix of W using $W^{1/2} = VD^{1/2}V^{-1}$.
Establish $D^{1/2} = \operatorname{diag}\big(\sqrt{2}, \sqrt{2}, \sqrt{8}\big) = \sqrt{2}\operatorname{diag}(1, 1, 2)$.

$$W^{1/2} = \frac{\sqrt{2}}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix} = \frac{\sqrt{2}}{3} \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

Part (d): Let $\bar{x}$ be a particular solution of the system $Ax = b$. Prove, using the transformation $y = W^{1/2}(x - \bar{x})$, that (P) is equivalent to the following:

$$(Q)\begin{cases}\min & \frac{1}{2}\|y\|^2 + \langle W^{1/2}\bar{x}, y\rangle \\ \text{s.t.} & AW^{-1/2}y = 0.\end{cases}$$

Let $\bar{x} = \begin{bmatrix}\bar{x}_1 & \bar{x}_2 & \bar{x}_3\end{bmatrix}^\mathsf{T}$ be a particular solution such that $A\bar{x} = b$.

$$y = W^{1/2}(x-\bar{x}) = \frac{\sqrt{2}}{3}\begin{bmatrix}4 & -1 & -1\\ -1 & 4 & 1\\ -1 & 1 & 4\end{bmatrix}\begin{bmatrix}x_1 - \bar{x}_1\\ x_2 - \bar{x}_2\\ x_3 - \bar{x}_3\end{bmatrix} = \frac{\sqrt{2}}{3}\begin{bmatrix}4(x_1 - \bar{x}_1) - (x_2 - \bar{x}_2) - (x_3 - \bar{x}_3)\\ -(x_1 - \bar{x}_1) + 4(x_2 - \bar{x}_2) + x_3 - \bar{x}_3\\ -(x_1 - \bar{x}_1) + x_2 - \bar{x}_2 + 4(x_3 - \bar{x}_3)\end{bmatrix}$$

Compute the normed expression:

$$\begin{aligned}\frac{1}{2}\|y\|^2 = \frac{1}{9}\big[&(4x_1 - 4\bar{x}_1 - x_2 + \bar{x}_2 - x_3 + \bar{x}_3)^2\\ &+ (-x_1 + \bar{x}_1 + 4x_2 - 4\bar{x}_2 + x_3 - \bar{x}_3)^2\\ &+ (-x_1 + \bar{x}_1 + x_2 - \bar{x}_2 + 4x_3 - 4\bar{x}_3)^2\big]\\ &= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_1x_3 + 2x_2x_3 + 2x_3^2 - 4x_1\bar{x}_1 + 2x_2\bar{x}_1 + 2x_3\bar{x}_1 + 2\bar{x}_1^2 + 2x_1\bar{x}_2\\ &- 4x_2\bar{x}_2 - 2x_3\bar{x}_2 - 2\bar{x}_1\bar{x}_2 + 2\bar{x}_2^2 + 2x_1\bar{x}_3 - 2x_2\bar{x}_3 - 4x_3\bar{x}_3 - 2\bar{x}_1\bar{x}_3 + 2\bar{x}_2\bar{x}_3 + 2\bar{x}_3^2\end{aligned}$$

Compute the inner product:

$$\begin{aligned}\langle W^{1/2}\bar{x}, y\rangle = \frac{\sqrt{2}}{3}\begin{bmatrix}4\bar{x}_1 - \bar{x}_2 - \bar{x}_3\\ -\bar{x}_1 + 4\bar{x}_2 + \bar{x}_3\\ \bar{x}_1 + \bar{x}_2 + 4\bar{x}_3\end{bmatrix} \cdot \frac{\sqrt{2}}{3}\begin{bmatrix}4x_1 - 4\bar{x}_1 - x_2 + \bar{x}_2 - x_3 + \bar{x}_3\\ -x_1 + \bar{x}_1 + 4x_2 - 4\bar{x}_2 + x_3 - \bar{x}_3\\ -x_1 + \bar{x}_1 + x_2 - \bar{x}_2 + 4x_3 - 4\bar{x}_3\end{bmatrix}\\ = 4x_1\bar{x}_1 - 2x_2\bar{x}_1 - 2x_3\bar{x}_1 - 4\bar{x}_1^2 - 2x_1\bar{x}_2\\ + 4x_2\bar{x}_2 + 2x_3\bar{x}_2 + 4\bar{x}_1\bar{x}_2 - 4\bar{x}_2^2 - 2x_1\bar{x}_3 + 2x_2\bar{x}_3 + 4x_3\bar{x}_3 + 4\bar{x}_1\bar{x}_3 - 4\bar{x}_2\bar{x}_3 - 4\bar{x}_3^2\end{aligned}$$

Add their expansions:

$$\begin{aligned}\frac{1}{2}\|y\|^2 + \langle W^{1/2}\bar{x}, y\rangle &= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_1x_3 + 2x_2x_3 + 2x_3^2\\ &- 2\bar{x}_1^2 + 2\bar{x}_1\bar{x}_2 - 2\bar{x}_2^2 + 2\bar{x}_1\bar{x}_3 - 2\bar{x}_2\bar{x}_3 - 2\bar{x}_3^2\\ &= (x_1 - x_2)^2 + (x_2 + x_3)^2 + (x_3 - x_1)^2\\ &- (\bar{x}_1 - \bar{x}_2)^2 - (\bar{x}_2 + \bar{x}_3)^2 - (\bar{x}_3 - \bar{x}_1)^2\\ &= f(x) - f(\bar{x})\end{aligned}$$

The constraint $AW^{-1/2}y = AW^{-1/2}W^{1/2}(x - \bar{x}) = Ax - A\bar{x} = 0 \iff Ax = A\bar{x} \iff Ax = b$. Here $f(\bar{x})$ is a constant ($\bar{x}$ is a particular value), thus $\arg\min_x[f(x) - f(\bar{x})] = \arg\min_x f(x)$.

The constraints and minimization in (P) and (Q) are equivalent, thus $(P) \iff (Q)$.

Part (e): Prove that (Q) has a unique solution that you compute.

A problem (R) of the form:

$$(R)\begin{cases}\min & \frac{1}{2}\|y\|^2 + \langle a, y\rangle \\ \text{s.t.} & By = 0\end{cases}$$

where $a \in \mathbb{R}^n$, $\dim B = m \times n$, and $\operatorname{rank}(B) = m$, has a unique solution y^* of the form:

$$y^* = \left[B^\mathsf{T}\left(BB^\mathsf{T}\right)^{-1}B - I_n\right]a.$$

In (Q), the conditions hold with $n = 3$ and $m = 2$ since the vector $a = W^{1/2}\bar{x} \in \mathbb{R}^3$, $B = AW^{-1/2}$, with $\dim(B) = 2 \times 3$, and $\operatorname{rank}(AW^{-1/2}) = \operatorname{rank}(A) = 2$ (since $W^{-1/2}$ is non-singular). Thus, (Q) has a unique solution y^* as above, with the appropriate substitutions.

Compute the matrix $W^{-1/2}$ using the Cofactor Method as in Part (c), and deduce B:

$$W^{-1/2} = \frac{\sqrt{2}}{12}\begin{bmatrix}5 & 1 & 1\\ 1 & 5 & -1\\ 1 & -1 & 5\end{bmatrix} \qquad B = AW^{-1/2} = \frac{\sqrt{2}}{12}\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}5 & 1 & 1\\ 1 & 5 & -1\\ 1 & -1 & 5\end{bmatrix} = \frac{\sqrt{2}}{6}\begin{bmatrix}5 & 4 & 7\\ 9 & 6 & 3\end{bmatrix}$$

Obtain a particular solution $\bar{x} = \begin{bmatrix}2 & 4 & 0\end{bmatrix}^\mathsf{T}$ by solving $Ax = b$ with $x_3 = 0$.
Compute the vector a,

$$a = W^{1/2}\bar{x} = \frac{\sqrt{2}}{3}\begin{bmatrix}4 & -1 & -1\\ -1 & 4 & 1\\ -1 & 1 & 4\end{bmatrix}\begin{bmatrix}2\\4\\0\end{bmatrix} = \frac{\sqrt{2}}{3}\begin{bmatrix}4\\14\\2\end{bmatrix}$$

Compute the product BB^T and its inverse,

$$BB^\mathsf{T} = \left(\frac{\sqrt{2}}{6}\right)^2\begin{bmatrix}5 & 4 & 7\\ 9 & 6 & 3\end{bmatrix}\begin{bmatrix}5 & 9\\ 4 & 6\\ 7 & 3\end{bmatrix} = \begin{bmatrix}5 & 5\\ 5 & 7\end{bmatrix} \Longrightarrow \left(BB^\mathsf{T}\right)^{-1} = \frac{1}{10}\begin{bmatrix}7 & -5\\ -5 & 5\end{bmatrix}.$$

Compute the expression of matrices in the explicit solution of y^*,

$$B^\mathsf{T}\left(BB^\mathsf{T}\right)^{-1}B - I_3 = \frac{1}{10}\left(\frac{\sqrt{2}}{6}\right)^2\begin{bmatrix}5 & 9\\ 4 & 6\\ 7 & 3\end{bmatrix}\begin{bmatrix}7 & -5\\ -5 & 5\end{bmatrix}\begin{bmatrix}5 & 4 & 7\\ 9 & 6 & 3\end{bmatrix} - I_3 = \frac{1}{90}\begin{bmatrix}-25 & 40 & 5\\ 40 & -64 & 8\\ -5 & 8 & -1\end{bmatrix}.$$

Obtain the solution y^* (with which the minimum value in (Q) could easily be obtained),

$$y^* = \frac{\sqrt{2}}{3 \times 90}\begin{bmatrix}-25 & 40 & 5\\ 40 & -64 & 8\\ -5 & 8 & -1\end{bmatrix}\begin{bmatrix}4\\12\\2\end{bmatrix} = \frac{\sqrt{2}}{3}\begin{bmatrix}5\\-8\\1\end{bmatrix}.$$

Part (f): Deduce that (P) has a unique solution that you compute.

Since $y = W^{1/2}(x - \bar{x})$ then $x^* = \bar{x} + W^{-1/2}y^*$.

$$x^* = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + \frac{\sqrt{2} \times \sqrt{2}}{12 \times 3} \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Finally, the value for which (P) is optimal is,

$$x^* = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

And with $x_1^* = 3, x_2^* = 2, x_3^* = 1$, the minimum $f(x^*)$ is,

$$\begin{aligned} f(x^*) &= (3-2)^2 + (2+1)^2 + (1-3)^2 \\ &= 1 + 9 + 4 \\ &= 14 \end{aligned}$$