# Solution to "Sequential Compactness"

Let $(E, d)$ be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $E$ that converges to $a \in E$. Prove that the set $\{x_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.

**Proof:**
Let $a = x_\infty$ and $S = \{x_n : n \in \mathbb{N}\} \cup \{x_\infty\}$.

**Case 1:** $(x_n)$ has finitely many values.

$$S \text{ is finite} \implies S \text{ is compact}$$

**Case 2:** $(x_n)$ has infinitely many values.
Let $(y_k)_{k\in\mathbb{N}} \in S$ be an arbitrary sequence.

$(y_k)$ is not necessarily a subsequence of $(x_n)$, does not necessarily contain all the terms of $(x_n)$, and thus can be **any** reordering of **some** terms of $(x_n)$.

Construct a subsequence $(y_{k_i})_{i\in\mathbb{N}}$ of $(y_k)$ inductively using the following algorithm:

$$\text{Step 0:} \begin{cases} N_0 = \big\{n \in \mathbb{N} : x_n \in (y_k)\big\} \\ n_0 = \min\{N_0\} \\ y_{k_0} = x_{n_0} \end{cases}$$

$$\text{Step 1:} \begin{cases} N_1 = \big\{n \in \mathbb{N} : x_n \in (y_k), n > n_0\big\} = N_0 \cap \big\{n \in \mathbb{N} : n > n_0\big\} \\ n_1 = \min\{N_1\} \\ y_{k_1} = x_{n_1} \end{cases}$$

$$\text{Step } i\text{:} \begin{cases} N_i = \big\{n \in \mathbb{N} : x_n \in (y_k), n > n_{i-1}\big\} = N_{i-1} \cap \big\{n \in \mathbb{N} : n > n_{i-1}\big\} \\ n_i = \min\{N_i\} \\ y_{k_i} = x_{n_i} \end{cases}$$

**Example:** If $(y_k) = (x_9, x_2, x_0, x_5, x_1, x_7, x_3, ...)$ then $(y_{k_i}) = (x_0, x_1, x_3, ...)$.

The resulting subsequence $(y_{k_i})$ of $(y_k)$ is also a subsequence of $(x_n)$.
Since $x_n \to x_\infty$ then $y_{k_i} \to x_\infty$.
Therefore $\forall y_k \in S, \exists y_{k_i} \to x_\infty \in S$.

Therefore $S$ is compact.