Solution to "Metric Topology"

Let $(\mathbb{R}, d)$ be the metric space where d is the usual metric, with $A \subset \mathbb{R}$ such that:

$$A = \left\{ (-1)^n + \frac{1}{n}, n \in \mathbb{N}^* \right\}$$

The below plot places the first 1000 elements of A on the real number line.

Interior Points

Since $\forall n \in \mathbb{N}^*$, $\big((-1)^n + 1/n\big) \in \mathbb{Q}$, then $A \subset \mathbb{Q}$.
Since $\operatorname{Int}(A) \subset \operatorname{Int}(\mathbb{Q}) = \varnothing$, therefore:

$$\operatorname{Int}(A) = \varnothing$$

Isolated Points

For all $n \in \mathbb{N}^*$, let:

$$N = \left((-1)^n + \frac{1}{n+1}, (-1)^n + \frac{1}{n-1} \right)$$

Since:

$$\forall a_n \in A, N \cap A = \{a_n\}$$

Therefore:

$$\operatorname{Iso}(A) = A$$

Accumulation Points

Since $A = \text{Iso}(A)$ then $\text{Acc}(A) \cap A = \varnothing$.

Since $\forall a_n \in A : -1 \leq a_n \leq 1.5$ then: $\forall x \in (-\infty, -1) \cup (1.5, \infty)$

$$\exists \rho > 0 : B(x, \rho) \cap A = \varnothing \implies x \notin \text{Acc}(A)$$

The sequence (a_n) can be split into two cases: $n = 2k$ and $n = 2k - 1$

$$a_n = \begin{cases} a_{2k} = 1 + \dfrac{1}{2k} \\ a_{2k-1} = -1 + \dfrac{1}{2k-1} \end{cases} \forall k \in \mathbb{N}^*$$

For each sub-sequence, define the open interval between two consecutive terms:

$$U = \ (a_{k+1}, a_k) \ = \begin{cases} U_{2k} = \ \left(1 + \dfrac{1}{2k+2}, 1 + \dfrac{1}{2k}\right) \\ U_{2k-1} = \ \left(-1 + \dfrac{1}{2k+1}, -1 + \dfrac{1}{2k-1}\right) \end{cases}$$

All of these intervals are, by definition, disjoint with A and are open, therefore: $\forall u \in \bigcup_{k=1}^{\infty} U$

$$\exists \rho > 0 : B(u, \rho) \cap A = \varnothing \implies u \notin \text{Acc}(A)$$

The only remaining points to test are $\{-1, 1\}$.
For $k \to \infty$: $a_{2k} \to 1$ and $a_{2k-1} \to -1$, therefore:

$$\forall \rho > 0 : \ (1 - \rho, 1 + \rho) \ \cap \ A - \{1\} \neq \varnothing$$

$$\forall \rho > 0 : \ (-1 - \rho, -1 + \rho) \ \cap \ A - \{-1\} \neq \varnothing$$

Therefore:

$$\text{Acc}(A) = \{-1, 1\}$$

Closure Points

Since $\text{Cl}(A) = \text{Acc}(A) \cup \text{Iso}(A)$, therefore:

$$\text{Cl}(A) = A \cup \{-1, 1\}$$