Solution to "Duality & Infimal–Convolution"

For f_1 and f_2 two proper functions from $\mathbb{R}^n \longrightarrow \mathbb{R} \cup \{\infty\}$, we define the *inf-convolution* of f_1 and f_2, denoted by $f_1 \square f_2$, is the function defined on $\mathbb{R}^n$ by:

$$(f_1 \square f_2)(x) := \inf_z \{f_1(z) + f_2(x-z)\}.$$

Part (a): Prove, using the definitions, that $(f_1 \square f_2)^* = f_1^* + f_2^*$.
By definition obtain, $(f_1 \square f_2)^*(y) = \sup_x \big\{\langle x, y\rangle - \inf_z\{f_1(z) + f_2(x-z)\}\big\}$.

Since $z + (x - z) = x$, then maximizing w.r.t. z is equivalent to minimizing w.r.t $x - z$ or x. Thus, $\inf_z\{f_1(z) + f_2(x-z)\} = \inf_x\{f_1(z) + f_2(x-z)\} = -\sup_x\{-f_1(z) - f_2(x-z)\}$.

Replace above to obtain $(f_1 \square f_2)^*(y) = \sup_x \big\{\langle x, y\rangle + \sup_x\{-f_1(z) - f_2(x-z)\}\big\}$. Simplify to get: $(f_1 \square f_2)^*(y) = \sup_x\{\langle x, y\rangle - f_1(z) - f_2(x-z)\} = \sup_x\{\langle z, y\rangle - f_1(z) + \langle x-z, y\rangle - f_2(x-z)\}$.

Since $x = z + (x - z)$, then maximizing w.r.t. x is equivalent to maximizing w.r.t z or $x - z$. Thus, $(f_1 \square f_2)^*(y) = \sup_{z,x-z}\{\underbrace{\langle z, y\rangle - f_1(z)}_{\phi(z)} + \underbrace{\langle x-z, y\rangle - f_2(x-z)}_{\psi(x-z)}\}$.

Since ϕ and ψ are maximized w.r.t. their respective variable, the supremum can be split. Obtain, $(f_1 \square f_2)^*(y) = \sup_z\{\langle z, y\rangle - f_1(z)\} + \sup_{x-z}\{\langle x-z, y\rangle - f_2(x-z)\} = f_1^*(y) + f_2^*(y)$.

Conclude $(f_1 \square f_2)^*(y) = (f_1^* + f_2^*)(y)$, as desired.

Part (b): Now let C be a nonempty closed and convex set and let $d_C(\cdot)$ be the distance function to the set C.

i. Verify that $d_C = \delta_C \square \|\cdot\|$.

Expand $\delta_C \square \|\cdot\|(x) = \inf_c\{\delta_C(c) + \|x - z\|\}$. Since only the case $c \in C$ is of interest, set the restriction and obtain that $\delta_C(c) = 0$. Thus obtain $\delta_C \square \|\cdot\|(x) = \inf_{c \in C} \|x - z\| = d_C(x)$.

ii. Deduce that $(d_C)^*(\cdot) = \sigma_C(\cdot) + \delta_{B_*}(\cdot)$, where $\sigma_C(\cdot)$ is the support of C and $\bar{B}_*$ is the unit closed ball for the dual norm.

Assert from known results that $(\delta_C)^*(\cdot) = \sigma_C(\cdot)$ and $(\|\cdot\|)^* = \delta_{B_*}(\cdot)$.
From Part (a), obtain $(d_C)^*(\cdot) = (\delta_C)^*(\cdot) + (\|\cdot\|)^* = \sigma_C(\cdot) + \delta_{B_*}(\cdot)$.

iii. Deduce that

$$\partial(d_C)^*(y) = \partial\sigma_C(y) + \partial\delta_{B_*}(y).$$

Notice that $\sigma_C(y)$ and $\delta_{B_*}(y)$ are closed and convex. Since δ_C and $\|\cdot\|$ are both clearly convex and proper, then conclude that $\sigma_C(y)$ and $\delta_{B_*}(y)$ are also proper.

Also note that $\operatorname{int}\big(\operatorname{dom}(\sigma_C)\big) \cap \operatorname{int}\big(\operatorname{dom}(\delta_{B_*})\big) = \mathbb{R}^n \cap \mathbb{R}^n = \mathbb{R}^n$.
These conditions permit the use of the Sum Rule: $\partial(f_1^* + f_2^*)(y) = \partial f_1^*(y) + \partial f_2^*(y),\ \forall y \in \mathbb{R}^n$.

Since $(d_C)^*(y) = \sigma_C(y) + \delta_{B_*}(y)$, obtain the desired result.

iv. Prove that $y \in \partial d_C(x)$ if and only if there exist $x_1 \in C$ and $x_2 \in \mathbb{R}^n$ such that

- $x = x_1 + x_2$
- $y \in \partial\delta_C(x_1)$ and $y \in \partial\|\cdot\|(x_2)$.

Since $\partial(d_C)^*(y) = \partial\sigma_C(y) + \partial\delta_{B_*}(y)$, then:
$x \in \partial(d_C)^*(y) \implies \exists x_1, x_2 : x_1 + x_2 = x$ and $(x_1, x_2) \in \partial\sigma_C(y) \times \partial\delta_{B_*}(y)$.

The converse of this statement is trivially obtained since it holds $\forall x_1, x_2 : x_1 + x_2 = x$ and $(x_1, x_2) \in \partial\sigma_C(y) \times \partial\delta_{B_*}(y)$. Thus conclude with the equivalence:

$$x \in \partial(d_C)^*(y) \iff \exists x_1, x_2 : x_1 + x_2 = x \text{ and } (x_1, x_2) \in \partial\sigma_C(y) \times \partial\delta_{B_*}(y) \tag{4.1}$$

Since d_C is convex and proper, then $(d)^*_C$ is proper. Having:
$(d_C)^*$, $\sigma_C = (\delta_C)^*$, and $\delta_{B_*} = (\|\cdot\|)^*$ all closed, convex and proper, apply the following:

$$\begin{aligned} x \in \partial(d_C)^*(y) &\iff y \in \partial d_C(x) \\ x_1 \in \partial\sigma_C(y) &\iff y \in \partial\delta_C(x_1) \\ x_2 \in \partial\delta_{B_*}(y) &\iff y \in \partial\|\cdot\|(x_2) \end{aligned}$$

Replace in (4.1) each statement with its equivalent statement as above:

$$y \in \partial d_C(x) \iff \exists x_1, x_2 : x_1 + x_2 = x \text{ and } y \in \partial\delta_C(x_1) \text{ and } y \in \partial\|\cdot\|(x_2)$$

Note that in both $(\Longrightarrow)$ and $(\Longleftarrow)$, $x_1 \in \operatorname{dom}(\delta_C) = C$ and $x_2 \in \operatorname{dom}(\|\cdot\|) = \mathbb{R}^n$ in order for $\partial\delta_C(x_1)$ and $\partial\|\cdot\|(x_2)$ to exist. Thus obtain:

$$y \in \partial d_C(x) \iff \exists x_1 \in C, x_2 \in \mathbb{R}^n : x_1 + x_2 = x \text{ and } y \in \partial\delta_C(x_1) \text{ and } y \in \partial\|\cdot\|(x_2) \tag{4.2}$$

As desired.

v. Deduce that

$$\partial d_C(x) = \begin{cases} N_C(x) \cap \bar{B} & \text{if } x \in C, \\ \dfrac{x - c_x}{\|x - c_x\|} & \text{if } x \notin C, \end{cases}$$

where $\bar{B}$ is the unit closed ball and c_x is the unique projection of x to the set C.

Rewrite (4.2) with $x_1 = c \in C$ and $x_2 = x - c \in \mathbb{R}^n$:

$$y \in \partial d_C(x) \iff y \in \partial \delta_C(c) \text{ and } y \in \partial \| \cdot \| (x - c) \tag{4.3}$$

***Case 1*:** $x \in C$
For $x = c \in C$, obtain $y \in \partial \delta_C(x)$ and $y \in \partial \| \cdot \|(0)$.
Realize that $\partial \delta_C(x) = N_C(x)$ (HW5.4: Case $x \in C$) and $\partial \| \cdot \|(0) = \bar{B}$.

Therefore $y \in N_C(x) \cap \bar{B}$ if $x \in C$, thus obtain from (4.3) that $\partial d_C(x) \subset N_C(x) \cap \bar{B}$ and $\partial d_C(x) \supset N_C(x) \cap \bar{B}$ if $x \in C$. Conclude that $\partial d_C(x) = N_C(x) \cap \bar{B}$ if $x \in C$.

***Case 2*:** $x \notin C$
For $x \notin C$, notice that $x \neq c \in C$ and $0 \neq x - c \in \mathbb{R}^n$:
Obtain $\partial \delta_C(c) = N_C(c)$ and $\partial \| \cdot \|(x - c) = \big\{(x - c)/\|x - c\|\big\}$.

Here $y \in N_C(c)$ and $y = (x - c)/\|x - c\|$. Since C is closed and convex and $c \in C$, then: $\langle y, c' - c \rangle \leq 0$, $\forall c' \in C$ and $\langle x - c_x, c - c_x \rangle \leq 0$.

Realize that $\langle y, c' - c \rangle \leq 0 \iff \langle x - c, c' - c \rangle \leq 0$ for all $c' \in C$.
Choose $c' = c_x$ and obtain $\langle x - c, c_x - c \rangle \leq 0$ and $\langle x - c_x, c - c_x \rangle \leq 0$

Addition: $\langle x - c, c_x - c \rangle + \langle x - c_x, c - c_x \rangle = \langle c_x - c, c_x - c \rangle = \|c - c_x\|^2 \leq 0 \iff c = c_x$.
Therefore, $y \in N_C(c)$ and $y = (x - c)/\|x - c\|$ gives $y = (x - c_x)/\|x - c_x\|$.

Thus obtain from (4.3) that $\partial d_C(x) \subset \big\{(x - c_x)/\|x - c_x\|\big\}$ and $\partial d_C(x) \supset \big\{(x - c_x)/\|x - c_x\|\big\}$ if $x \notin C$. Conclude that $\partial d_C(x) = \big\{(x - c_x)/\|x - c_x\|\big\}$ if $x \notin C$.

vi. Deduce that $d_C(\cdot)$ is differentiable at any $x \notin C$.

The function $d_C(\cdot)$ is clearly proper, lower-semicontinuous, and is convex. Since $x \notin C \implies x \in \text{int}(\text{dom}(d_C)) = \mathbb{R}^n$, then:

Having $\partial d_C(x) = \left\{ \dfrac{x - c_x}{\|x - c_x\|} \right\}$ for $x \notin C$, conclude $d_C(x)$ is differentiable for any $x \notin C$.