Solution to "Convexity of Minimization Problem"

For $a \in \mathbb{R}$, we consider the following minimization problem

$$(P_a) \begin{cases} \min\limits_{(x_1,x_2)\in\mathbb{R}^2} x_1^2 + ax_2^2 + x_1x_2 + x_1, \\ \text{s.t.} \quad x_1 + x_2 - 1 \leq 0. \end{cases}$$

Part (a): Prove that (P_a) is a convex problem if and only if $a \geq \frac{1}{4}$.

The problem (P_a) is convex if and only if $\mathcal{D}$ is convex, and the functions f_0, f_i, and h_i are all convex over $\mathcal{D}$. Here $\mathcal{D} = \mathbb{R}^2$ is clearly convex, and the constraint function $f_1(x_1, x_2) = x_1 + x_2 - 1$ is affine and is thus convex over $\mathbb{R}^2$.

Study the convexity of f_0 w.r.t. a to determine the range of a for which (P_a) is convex. Compute the Hessian matrix of f_0 and express its eigenvalues λ_1 and λ_2 as a function of a.

$$\nabla^2 f_0 = \begin{bmatrix} 2 & 1 \\ 1 & 2a \end{bmatrix} \Longrightarrow \det(\nabla^2 f_0 - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2a-\lambda \end{vmatrix} = \lambda^2 - (2a+2)\lambda + 4a - 1 = 0$$

The discriminant $\Delta = (2a+2)^2 - 16a + 4 = 4(a^2 - 2a + 2) = 4[(a-1)^2 + 1] > 0$ verifies that λ_1 and λ_2 are real for any a. Use the quadratic formula to obtain $\lambda_{1,2} = \frac{1}{2}(2a + 2 \mp \sqrt{\Delta})$. Thus obtain the solution:

$$\lambda_1 = a + 1 - \sqrt{(a-1)^2 + 1} \text{ and } \lambda_2 = a + 1 + \sqrt{(a-1)^2 + 1}$$

The system $\{\lambda_1 \geq 0, \lambda_2 \geq 0\}$ translates to $\left\{a+1 \geq \sqrt{(a-1)^2+1},\ a+1 \geq -\sqrt{(a-1)^2+1}\right\}$.

Since $a + 1 \geq \sqrt{(a-1)^2 + 1} \geq 0 \geq -\sqrt{(a-1)^2 + 1}$ the conjunction reduces the system to solving the following inequality:

$$a+1 \geq \sqrt{(a-1)^2+1} \Longleftrightarrow (a+1)^2 - (a-1)^2 \geq 1 \Longleftrightarrow (a+1+a-1)(a+1-a+1) \geq 1 \Longleftrightarrow 4a \geq 1$$

The following equivalences are established:

$$a \geq \frac{1}{4} \Longleftrightarrow \lambda_1, \lambda_2 \geq 0 \Longleftrightarrow \nabla^2 f_0 \text{ is positive semi-definite} \Longleftrightarrow f_0 \text{ is convex.}$$

Therefore, (P_a) is a convex problem if and only if $a \geq \frac{1}{4}$.

Part (b): Find the optimal solution and the optimal value of (P_a) when $a \geq \frac{1}{4}$.

Since $\exists(-1,-1) \in \text{int}(\mathbb{R}^2) : f_1(-1,-1) = -1-1-1 = -3 < 0$ then Slater's condition is satisfied. Furthermore, f_0 and f_1 are clearly differentiable and (P_a) is convex with $a \geq \frac{1}{4}$.

Compute the gradients: $\nabla f_0(x_1, x_2) = \begin{pmatrix} 2x_1 + x_2 + 1 \\ x_1 + 2ax_2 \end{pmatrix}$ and $\nabla f_1(x_1, x_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

x^* is optimal for $(P_a)_{a \geq \frac{1}{4}}$ if and only if $\exists \lambda_1^*$ that satisfies KKT:

$$(D_a)_{a \geq \frac{1}{4}} : \begin{cases} x_1^* + x_2^* - 1 \leq 0 \\ \lambda_1^* \geq 0 \\ \lambda_1^*(x_1^* + x_2^* - 1) = 0 \\ 2x_1^* + x_2^* + 1 + \lambda_1^* = 0 \\ x_1^* + 2ax_2^* + \lambda_1^* = 0 \end{cases}$$

***Case 1*:** $\lambda_1^* = 0$

The system $(D_a)_{a \geq \frac{1}{4}}$ reduces to solving: $\begin{cases} 2x_1^* + x_2^* + 1 = 0 \\ x_1^* + 2ax_2^* = 0 \end{cases} \iff \begin{cases} x_1^* = -\frac{2a}{4a-1} \\ x_2^* = \frac{1}{4a-1} \end{cases}$

Check (x_1^*, x_2^*) feasibility range: $x_1^* + x_2^* - 1 = \frac{2(1-3a)}{4a-1} \leq 0$ for $a \geq \frac{1}{3}$.
Substitute above and simplify to obtain the optimal value $f_0(x_1^*, x_2^*) = \frac{a}{1-4a}$ for $a \geq \frac{1}{3}$.

***Case 2*:** $x_1^* + x_2^* - 1 \iff x_1 = 1 - x_2$

The system $(D_a)_{a \geq \frac{1}{4}}$ reduces to solving: $\begin{cases} -x_2^* + 3 + \lambda^* = 0 \\ (2a-1)x_2^* + 1 + \lambda^* = 0 \end{cases} \iff \begin{cases} x_1^* = 1 - \frac{1}{a} \\ x_2^* = \frac{1}{a} \\ \lambda^* = \frac{1-3a}{a} \end{cases}$

Check λ^* feasibility range: $\lambda^* = \frac{1-3a}{a} \geq 0$ for $a \in \left[\frac{1}{4}, \frac{1}{3}\right]$.
Substitute above and simplify to obtain the optimal value $f_0(x_1^*, x_2^*) = 2 - \frac{1}{a}$ for for $a \in \left[\frac{1}{4}, \frac{1}{3}\right]$.

***Conclusion*:**
The optimal solutions of $(P_a)_{a \geq \frac{1}{4}}$ and their optimal values are:

$$\begin{cases} x_1^* = 1 - \dfrac{1}{a} \text{ and } x_2^* = \dfrac{1}{a} \text{ and } f_0^* = 2 - \dfrac{1}{a} & \text{for } \dfrac{1}{4} \leq a \leq \dfrac{1}{3} \\ \\ x_1^* = -\dfrac{2a}{4a-1} \text{ and } x_2^* = \dfrac{1}{4a-1} \text{ and } f_0^* = \dfrac{a}{1-4a} & \text{for } a \geq \dfrac{1}{3} \end{cases}$$

Interestingly for $a = \frac{1}{3}$ the two minimal points are a double-point $(-2, 3)$ for which $f^* = -1$.