Solution to "Semicontinuity"

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a lower semicontinuous function, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a lower semicontinuous and nondecreasing function. Show that the function h defined by $h(x) = g\big(f(x)\big)$ is lower semicontinuous. Give an example showing that the nondecrease assumption is essential.

Proof:
It is sufficient to prove that for all $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^n$ with $x_n \to x_\infty$,

$$\liminf_{n\to\infty} g\big(f(x_n)\big) \geq g\big(f(x_\infty)\big). \tag{2.1}$$

Let $x_n \to x_\infty$ be any sequence in $\mathbb{R}^n$.
Since f is lower semicontinuous, then $\liminf_{n\to\infty} f(x_n) \geq f(x_\infty)$, and since g is nondecreasing then:

$$g\Big(\liminf_{n\to\infty} f(x_n)\Big) \geq g\big(f(x_\infty)\big) \tag{2.2}$$

The limit inferior of $f(x_n)$ is by definition:

$$\lim_{n\to\infty}\left(\inf_{k\geq n} f(x_k)\right) = \liminf_{n\to\infty} f(x_n) \tag{2.3}$$

Interpret (2.3) as a sequence $\inf_{k\geq n} f(x_k) \to \liminf_{n\to\infty}\big(f(x_n)\big)$. Since g is lower semicontinuous then:

$$\liminf_{n\to\infty} g\Big(\inf_{k\geq n} f(x_k)\Big) \geq g\Big(\liminf_{n\to\infty} f(x_n)\Big) \tag{2.4}$$

Since $f(x_n) \geq \inf_{k\geq n} f(x_k)$ and g is nondecreasing then $g\big(f(x_n)\big) \geq g\big(\inf_{k\geq n} f(x_k)\big)$. Thus,

$$\liminf_{n\to\infty} g\big(f(x_n)\big) \geq \liminf_{n\to\infty} g\Big(\inf_{k\geq n} f(x_k)\Big). \tag{2.5}$$

Combine (2.2), (2.4), and (2.5) to obtain (2.1), as desired.

Example:
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ with:

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = -x$$

Let $(x_n)_{n\in\mathbb{N}^*} = (1/n)_{n\in\mathbb{N}^*} > 0$ with $x_n \to 0$.

Both f and g are lower semicontinuous since $\liminf_{n\to\infty} f(x_n) = \liminf_{x\to\infty} 1 = 1 \geq f(0) = 0$ and g is continuous. However, $\liminf_{n\to\infty} g\big(f(x_n)\big) = \liminf_{n\to\infty} -1 = -1 > g\big(f(0)\big) = 0$. Thus, $h = g \circ f$ is not lower semicontiuous.

Therefore, g is necessarily nondecreasing for $g \circ f$ to be lower semicontinuous.